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Massive particle production in anisotropic space–times

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Abstract. Using a perturbation expansion to second order in a quantity determining deviation from conformal invariance, we obtain general expressions for the number and energy densities at late times of free, massive, scalar particles produced by quantum effects in an anisotropic space–time. Some interesting special cases are given and cosmological implications discussed.

1. Introduction

The computational difficulties involved in obtaining exact results from quantum field theory in curved space–time have led to the calculation of particle production (Zeldovich and Starobinski 1977, Fischetti and Hartle 1978, Birrell 1979) and stress tensors (Davies and Unruh 1979, Fischetti *et al* 1979, Hartle and Hu 1979, Horowitz 1979) in perturbation theory in the metric about flat space–time. The power of this technique is demonstrated here as we extend some of the previous results to massive particles and non-isotropic space–times, calculating the number and energy density of particles created by a small gravitational disturbance. The expressions, which are non-local as expected, apply to late times, when the disturbance has ceased. The computation of stress tensors which apply at all times is much more complicated.

Davies and Unruh (1979) obtained a concrete expression for $\langle T_{\mu\nu}\rangle$, the vacuum expectation value of the stress tensor of a massless, non-conformally coupled scalar field ϕ propagating in a spatially flat Robertson–Walker space–time. At late times, where the space–time curvature can be neglected (e.g. if the metric approaches that of Minkowski space or a radiation-dominated Friedmann universe for which $R=0$), the only surviving term of their expression is

$$\langle T_{\mu\nu}\rangle = -\frac{1}{32\pi}\epsilon_{\mu\nu}a^{-2}(\eta)\int_{-\infty}^{\infty}d\eta_1\int_{-\infty}^{\infty}d\eta_2\, g'(\eta_1)g'(\eta_2)\ln|\eta_1-\eta_2| \tag{1.1}$$

where a is the conformal scale factor, η is the conformal time parameter, $g=(\xi-\frac{1}{6})a^2R$, with ξ the conformal coupling constant of the field, and $\epsilon_{\mu\nu}=\mathrm{diag}(1,\frac{1}{3},\frac{1}{3},\frac{1}{3})$ in Minkowski coordinates. The created particles thus behave like massless radiation with an energy per unit volume

$$\rho = -\frac{1}{32\pi a^4}\int_{-\infty}^{\infty}d\eta_1\int_{-\infty}^{\infty}d\eta_2\, g'(\eta_1)g'(\eta_2)\ln|\eta_1-\eta_2|. \tag{1.2}$$

In what follows we shall generalise (1.2) to the case of a massive field and an anisotropic space–time.

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The field satisfies

$$(\Box + m^2 + \xi R)\phi = 0$$

and the background space–time is taken to have the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3} (1 + h_i(\eta))\, dx_i^2\right) \tag{1.3}$$

where it is assumed that

$$\max|h_i(\eta)| \ll 1, \qquad i = 1, 2, 3. \tag{1.4}$$

In this space–time the field ϕ possesses mode solutions of the form

$$\phi_k(x) = (2\omega)^{-1/2} a^{-1}\, e^{i\mathbf{k}.\mathbf{x}} \psi_k(\eta) \tag{1.5}$$

where

$$\psi_k(\eta) = e^{-i\omega\eta} + \frac{1}{\omega}\int_{-\infty}^{\eta} V_k(\eta_1) \sin \omega(\eta - \eta_1)\psi_k(\eta_1)\, d\eta_1 \tag{1.6}$$

with

$$V_k(\eta) = \sum_i h_i(\eta) k_i^2 + m^2(a^2(\infty) - a^2(\eta)) - (\xi - \tfrac{1}{6})R(\eta)a^2(\eta) \tag{1.7}$$

and

$$\omega^2 = k^2 + m^2 a^2(\infty). \tag{1.8}$$

This simple solution only results if we restrict the h_i by the condition

$$\sum_i h_i(\eta) = 0. \tag{1.9}$$

In the late time region ($\eta \to \infty$) (1.6) gives

$$\psi_k(\eta) = \alpha_\omega\, e^{-i\omega\eta} + \beta_\omega\, e^{i\omega\eta} \tag{1.10}$$

where the Bogolubov coefficients are given by

$$\alpha_\omega = 1 + \frac{i}{2\omega}\int_{-\infty}^{\infty} e^{i\omega\eta} V_k(\eta)\psi_k(\eta)\, d\eta, \tag{1.11}$$

$$\beta_\omega = -\frac{i}{2\omega}\int_{-\infty}^{\infty} e^{-i\omega\eta} V_k(\eta)\psi_k(\eta)\, d\eta. \tag{1.12}$$

If we treat $V_k(\eta)$ as small, then (1.10)–(1.12) can be solved by iteration. To lowest order, (1.11) gives $\alpha_\omega = 1$, and (1.12) gives $\beta_\omega = 0$, which, in (1.10), give $\psi_k(\eta) = e^{-i\omega\eta}$. Substituting this approximation to ψ_k into (1.11) and (1.12) gives the Bogolubov coefficients to first order in V_k:

$$\alpha_\omega = 1 + \frac{i}{2\omega}\int_{-\infty}^{\infty} V_k(\eta)\, d\eta, \tag{1.13}$$

$$\beta_\omega = -\frac{i}{2\omega}\int_{-\infty}^{\infty} e^{-2i\omega\eta} V_k(\eta)\, d\eta. \tag{1.14}$$

If the quantum state chosen corresponds to the 'in' vacuum, then in the 'out' region ($\eta \to \infty$) the number density (per unit proper volume) is

$$n = \frac{1}{(2\pi a)^3} \int |\beta_\omega|^2 \, d^3k \tag{1.15}$$

and the energy density is

$$\rho = \frac{1}{(2\pi)^3 a^4} \int |\beta_\omega|^2 \omega \, d^3k. \tag{1.16}$$

For this approach to be valid, we must have $h_i(\eta)$, $a^2(\eta)R(\eta) \to 0$ as $\eta \to \pm\infty$ and in the massive case, in addition, $a(\eta) \to a(\infty) < \infty$ as $\eta \to \pm\infty$. The formalism can also be used for cosmological models which expand from a singularity at $\eta = 0$, so long as $V_k(\eta)$ vanishes as $\eta \to 0$. One simply replaces the lower limits of the η-integrations by zero.

2. Particle number

In this section the number of created particles per unit volume will be calculated for the space–time (1.3), under the conditions of the previous section.

First (1.14) is substituted into (1.15):

$$n = \frac{1}{32\pi^3 a^3} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \int \frac{d^3k}{\omega^2} \Big[\sum_i \sum_j (k_i^2 k_j^2 h_i(\eta_1) h_j(\eta_2)) + V(\eta_1) V(\eta_2) \Big] \exp[2i\omega(\eta_1 - \eta_2)], \tag{2.1}$$

where the isotropic part of the perturbation has been written as $V(\eta)$:

$$V(\eta) \equiv m^2(a^2(\infty) - a^2(\eta)) - (\xi - \tfrac{1}{6}) R(\eta) a^2(\eta). \tag{2.2}$$

Note that no single summation cross terms appear in (2.1) due to the fact that, under angular integration in k space,

$$\sum_i h_i \int k_i^2 \, d\hat{k} = \frac{4\pi}{3} k^2 \sum_i h_i = 0.$$

Therefore, to second order in the potential, the total particle number density is just the sum of the separate contributions from the anisotropic and isotropic perturbations.

The other angular integrals give 4π for the isotropic part and

$$\int k_i^2 k_j^2 \, d\hat{k} = \frac{4\pi}{5} k^4 \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \tag{2.3}$$

for the anisotropic part. Using the property (1.9) once again we have

$$\sum_i \sum_j \int k_i^2 k_j^2 h_i(\eta_1) h_j(\eta_2) \, d\hat{k} = \frac{8\pi}{15} k^4 \sum_i h_i(\eta_1) h_i(\eta_2). \tag{2.4}$$

To obtain a convergent answer, we must assume that h_i', $h_i'' \to 0$ as $\eta \to \pm\infty$, where the prime denotes differentiation with respect to η. Then integrating the Fourier transform of h_i twice by parts enables us to replace $h_i(\eta)$ by $-h_i''(\eta)/4\omega^2$. We can also replace the exponential in (2.1) by its real part, as the double η-integrations have the effect of symmetrising on $\eta_1-\eta_2$. Equation (2.1) then reduces to

$$n=\frac{1}{960\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\int_{-\infty}^{\infty}\mathrm{d}\eta_2\Big[\sum_i h_i''(\eta_1)h_i''(\eta_2)\int_0^{\infty}\Big(\frac{k}{\omega}\Big)^6\cos 2\omega(\eta_1-\eta_2)\,\mathrm{d}k$$
$$+120V(\eta_1)V(\eta_2)\int_0^{\infty}\Big(\frac{k}{\omega}\Big)^2\cos 2\omega(\eta_1-\eta_2)\,\mathrm{d}k\Big]. \qquad (2.5)$$

The k-integrations can be performed using an equation from Oberhettinger (1957, p 7) and the result derived in the Appendix to give

$$n=\frac{1}{960\pi a^3}\int_{-\infty}^{\infty}a^4(C^2(\eta)+60V^2(\eta))\,\mathrm{d}\eta$$
$$+\frac{\bar m}{960a^3}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\int_{-\infty}^{\infty}\mathrm{d}\eta_2\Big\{F[\bar m(\eta_1-\eta_2)]\Big[-60V(\eta_1)V(\eta_2)$$
$$+(8\bar m^4-6\bar m^2\,\partial_1\,\partial_2+\tfrac{3}{2}\partial_1^2\,\partial_2^2)\sum_i h_i(\eta_1)h_i(\eta_2)\Big]$$
$$-\mathrm{J}_1[2\bar m(\eta_1-\eta_2)]\Big(60V(\eta_1)V(\eta_2)+\tfrac{1}{2}\sum_i h_i''(\eta_1)h_i''(\eta_2)\Big)\Big\}, \qquad (2.6)$$

where $\partial_1\equiv\partial/\partial\eta_1$, etc, $C^2\equiv C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}=\frac{1}{2}\Sigma_i\,(h_i'')^2$, $\bar m\equiv ma(\infty)$ and

$$F(x)\equiv-1/\pi+x(\mathrm{J}_0(2x)\mathrm{H}_{-1}(2x)+\mathrm{H}_0(2x)\mathrm{J}_1(2x)), \qquad (2.7)$$

H_ν being Struve functions, J_ν Bessel functions.

Note that, in the massless limit, the first term of (2.6) survives. This limit agrees with the results of Zeldovich and Starobinski (1977).

3. The energy density

To compute ρ we can use expression (2.5) with an additional factor of ω/a in the integrand of the k-integrals. This necessitates a further integration by parts, resulting in factors of h_i''', which we also assume vanish at $\eta=\pm\infty$.

After performing the k-integrations one obtains

$$\rho=\frac{1}{3840\pi^2a^4}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\int_{-\infty}^{\infty}\mathrm{d}\eta_2\,\mathrm{Re}\,\mathrm{K}_0[2\mathrm{i}\bar m(\eta_1-\eta_2)](\partial_1\,\partial_2-4\bar m^2)\Big[120V(\eta_1)V(\eta_2)$$
$$+(\partial_1\,\partial_2-4\bar m^2)^2\sum_i h_i(\eta_1)h_i(\eta_2)\Big], \qquad (3.1)$$

where K_0 is a modified Bessel function of the second kind. Curiously, this expression is considerably simpler than (2.6).

In the massless limit, (3.1) reduces to

$$\rho=-\frac{1}{3840\pi^2a^4}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\int_{-\infty}^{\infty}\mathrm{d}\eta_2\ln[2\mathrm{i}\bar m(\eta_1-\eta_2)]\Big(120V'(\eta_1)V'(\eta_2)$$
$$+\sum_i h_i'''(\eta_1)h_i'''(\eta_2)\Big). \qquad (3.2)$$

If additionally $h_i = 0$, then, noting that the constant factor in the argument of the logarithm can be changed arbitrarily without affecting the result (since $\int_{-\infty}^{\infty} d\eta\, V'(\eta) = \int_{-\infty}^{\infty} d\eta\, h_i''(\eta) = 0$), (3.2) reduces to (1.1).

4. Explicit examples

Unfortunately there are very few choices of $h_i(\eta)$ or $V(\eta)$ for which the integrals in (2.6) or (3.1) can be performed in terms of known functions ($e^{-\alpha|\eta|}$, α constant, is one of the few). Nevertheless, the expressions are of great value for numerical computations.

To obtain some idea of the predictive power of the perturbation technique, we shall examine some explicit examples for which, by returning to (1.13)–(1.16) and performing the η-integrations first, closed-form answers may be obtained.

4.1. Isotropic space–time, conformally coupled massive field

Consideration of a conformally coupled ($\xi = \frac{1}{6}$) field in an isotropic space–time ($h_i = 0$) with scale factor

$$a^2(\eta) = 1 - A \exp(-\alpha^2\eta^2), \qquad \alpha, A \text{ constant}, \tag{4.1}$$

corresponding to a universe which contracts rapidly, bounces at $\eta = 0$ and expands out again symmetrically, enables both n and ρ to be evaluated. From (2.2) we have

$$V(\eta) = m^2 A \exp(-\alpha^2\eta^2) \tag{4.2}$$

and we find, using (1.14),

$$\beta_\omega = -\frac{i m^2 A \sqrt{\pi}}{2\omega\alpha} \exp(-\omega^2/\alpha^2), \tag{4.3}$$

where, from (1.8),

$$\omega^2 = k^2 + m^2. \tag{4.4}$$

Substitution into (1.15) yields standard integrals which give

$$n = \frac{m^4 A^2}{16\alpha}\left(\frac{\exp(-2m^2/\alpha^2)}{\sqrt{2\pi}} - \frac{m}{\alpha}(1 - \Phi(m\sqrt{2}/\alpha))\right), \tag{4.5}$$

where Φ is an error function.

Similarly, substitution into (1.16) yields

$$\rho = \frac{m^4 A^2 \exp(-2m^2/\alpha^2)}{8\pi\alpha^2} \int_0^\infty \frac{k^2 \exp(-2k^2/\alpha^2)}{(k^2+m^2)^{1/2}}\, dk.$$

This integral may be evaluated by using the following integral representation for $1/(k^2+m^2)^{1/2}$:

$$\frac{1}{(k^2+m^2)^{1/2}} = \frac{2}{\pi}\int_0^\infty K_0(mx) \cos kx\, dk. \tag{4.6}$$

By interchanging the order of integration, the first integral is now elementary and the second is K_0 multiplied by a Gaussian function, which can be performed in terms of a

Whittaker function. We obtain

$$\rho = \frac{m^4 A^2}{128\pi} \exp(-m^2/\alpha^2)\left[2K_0\left(\frac{m^2}{\alpha^2}\right) - \left(\frac{\pi}{2}\right)^{1/2} \frac{\alpha}{m} W_{-1,0}\left(\frac{2m^2}{\alpha^2}\right)\right]. \tag{4.7}$$

A useful check is that $n, \rho \to 0$ as $m \to 0$. Note also that $n, \rho \to 0$ as $m \to \infty$. These limits are approached as the dimensionless quantity m/α satisfies $m/\alpha \ll 1$ and $m/\alpha \gg 1$ respectively. The functions peak around $m \simeq \alpha$, which is expected on physical grounds as a sort of 'resonance', when the background space–time is changing on a timescale comparable with the particle Compton time. This behaviour has been verified numerically without the use of perturbation theory, using the momentum space method of Birrell (1979). Assuming that most particles are created in a time $\sim m^{-1}$, we may use this special case to draw the general conclusion that a Friedmann universe which expands from the Planck era with a decelerating rate (e.g. $a(t) \propto t^{1/2} \propto \eta$) will not produce conformally coupled particles appreciably until $\dot{a}/a \sim m$, i.e. $t \sim m^{-1}$.

4.2. Isotropic space–time, non-conformally coupled massive field

The result of § 4.1 suggests that if particle creation results from conformal symmetry breaking caused by the presence of a mass, it is relatively inefficient, as production is inhibited when the space–time varies rapidly compared to m^{-1}. If the conformal symmetry is instead broken by non-conformal coupling ($\xi \neq \frac{1}{6}$) then there is no length scale in the theory, and this suppression will not occur. We therefore expect that when the field is both massive and non-conformally coupled, the influence of the mass will be negligible in a realistic cosmological model.

To check this expectation we treat the model

$$a(\eta) = 1 - \alpha^2/2(\alpha^2 + \eta^2), \qquad \alpha \text{ constant}, \tag{4.8}$$

which, once again, corresponds to a universe which contracts to a small value of the scale factor, bounces, and expands out again in a symmetric fashion. This time, however, we assume $\xi \neq \frac{1}{6}$.

Noting that

$$R = 6(3\alpha^2\eta^2 - \alpha^4)/(\eta^2 + \alpha^2/2)^3, \tag{4.9}$$

one obtains

$$\beta_\omega = \frac{i\pi}{16\omega\alpha}[(A\omega + B)\,e^{-2\alpha\omega} - C\,e^{-\sqrt{2}\,\alpha\omega}] \tag{4.10}$$

where

$$A = 2\alpha^3 m^2 + 384\alpha\Lambda, \qquad B = 672\Lambda - 7\alpha^2 m^2,$$
$$C = 480\sqrt{2}\,\Lambda, \qquad \Lambda = (\xi - \tfrac{1}{6}). \tag{4.11}$$

Substitution of (4.10) into (1.16) yields standard k-integrals and we obtain

$$\rho = \frac{m}{512\alpha^2}\left[\frac{A^2}{64}\frac{\partial^2}{\partial\alpha^2}\left(\frac{K_1(4m\alpha)}{\alpha}\right) + A\frac{\partial}{\partial\alpha}\left(\frac{C}{(3+2\sqrt{2})\alpha}K_1[(2+\sqrt{2})m\alpha] - \frac{BK_1(4m\alpha)}{8\alpha}\right)\right.$$
$$\left. + \frac{B^2}{4\alpha}K_1(4m\alpha) + \frac{C^2}{2\sqrt{2}\,\alpha}K_1(2\sqrt{2}\,m\alpha) - \frac{2BC}{(2+\sqrt{2})\alpha}K_1[(2+\sqrt{2})m\alpha]\right]. \tag{4.12}$$

Note that as $m \to \infty$, $\rho \to 0$ as expected. However, in the massless limit, (4.12) does not vanish for non-zero Λ, but yields

$$\rho \simeq 1{\cdot}843\Lambda^2/\alpha^4 \simeq \Lambda^2 R^2_{\max}, \tag{4.13}$$

where $R_{\max}$ is the maximum value of the curvature. This accords well with the general results of Birrell *et al* (1980).

The dependence of $\alpha^4\rho$ on $m\alpha$ is shown in figure 1(*a*) for conformal coupling, $\xi = \frac{1}{6}$, and in figure 1(*b*) for minimal coupling, $\xi = 0$. As in § 4.1, the density in the conformally coupled case is distinctly peaked around $\alpha \simeq m^{-1}$, showing that maximum particle creation occurs when the expansion rate is comparable with the particle's Compton time. On the other hand, in the minimally coupled case, ρ is a very slowly varying function of m (for fixed α). Thus, the presence of a mass does not appreciably improve the particle production rate when the field is already non-conformally coupled.

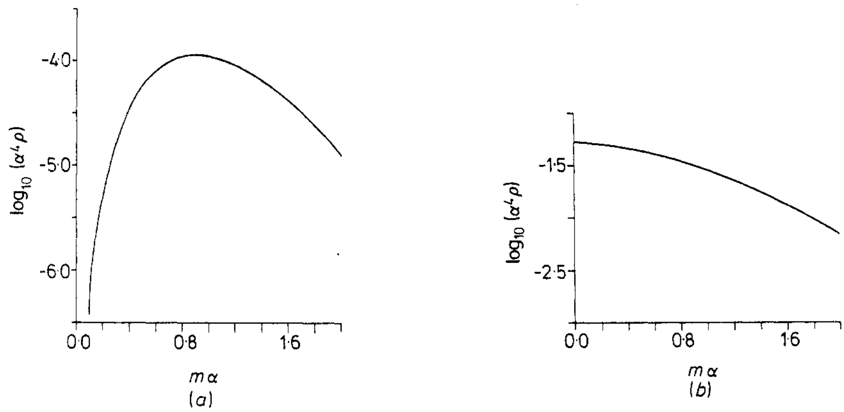

Figure 1. The energy density of created particles as a function of particle mass for a universe with scale factor (4.8). (*a*) is for conformally coupled particles, $\xi = \frac{1}{6}$, and (*b*) is for minimal coupling $\xi = 0$.

4.3. *Anisotropic space-time, massive field*

The conformal symmetry breaking that leads to particle production can arise in a third way, namely by departure of the background space-time from conformal flatness.

Consider first the space-time with metric (1.3) with

$$h_i(\eta) = \exp(-\alpha\eta^2)\cos(\beta\eta^2 + \delta_i) \tag{4.14}$$

where the δ_i differ by $2\pi/3$ so that $\Sigma_i h_i = 0$. This metric represents anisotropic oscillations of a general Friedmann space-time with scale factor $a(\eta)$. The space-time is reminiscent of the mixmaster moαel.

From (1.7) and (1.14), the contribution to the Bogolubov coefficient β_ω from the anisotropic perturbation is

$$\beta_\omega = -\frac{\mathrm{i}\sqrt{\pi}}{2\omega}\sum_i k_i^2 \operatorname{Re}\left(\frac{\exp[-\omega^2/(\alpha + \mathrm{i}\beta)]}{(\alpha + \mathrm{i}\beta)^{1/2}} \exp(-\mathrm{i}\delta_i)\right). \tag{4.15}$$

Substituting into (1.16) and using (2.3) one obtains

$$\rho = \frac{m^2 a^2(\infty)}{1536\sqrt{\pi}\, a^4} \frac{(\alpha^2+\beta^2)^{3/2}}{\alpha^2} \exp[-3\alpha m^2 a^2(\infty)/(\alpha^2+\beta^2)] W_{-\frac{3}{2},\frac{3}{2}}\left(\frac{2\alpha m^2 a^2(\infty)}{\alpha^2+\beta^2}\right) \tag{4.16}$$

for the contribution of the anisotropic perturbations to the energy density. In the massless limit this reduces to

$$\rho = \frac{1}{2880\pi} \frac{(\alpha^2+\beta^2)^{5/2}}{\alpha^3 a^4}. \tag{4.17}$$

As a second similar example consider

$$h_i(\eta) = \frac{\alpha^2}{\alpha^2+\eta^2} \cos(\beta\eta + \delta_i). \tag{4.18}$$

Then

$$\beta_\omega = -\frac{\mathrm{i}\alpha\pi}{4\omega} \sum_i k_i^2 (\mathrm{e}^{\mathrm{i}\delta_i}\, \mathrm{e}^{-|\alpha(2\omega-\beta)|} + \mathrm{e}^{-\mathrm{i}\delta_i}\, \mathrm{e}^{-\alpha(2\omega+\beta)}) \tag{4.19}$$

and

$$\rho = \frac{\alpha^2 \cosh 2\alpha\beta}{120 a^4} \int_0^\infty \theta(\omega-\beta/2) \frac{k^6}{\omega} \mathrm{e}^{-4\alpha\omega}\, \mathrm{d}k + \frac{\alpha^2\, \mathrm{e}^{-2\alpha\beta}}{120 a^4} \int_0^\infty \theta(\beta/2-\omega) \frac{k^6}{\omega} \cosh 4\alpha\omega\, \mathrm{d}k. \tag{4.20}$$

For $\beta < 2ma(\infty)$ only the first integral contributes and one finds

$$\rho = \frac{m^3 a^3(\infty) \cosh 2\alpha\beta}{512\alpha a^4} \mathrm{K}_3[4ma(\infty)\alpha]. \tag{4.21}$$

For $\beta > 2m$ the integrals cannot be performed in terms of known functions; however, in the massless limit, we find

$$\rho = [4(\alpha\beta)^5 + 20(\alpha\beta)^3 + 3\alpha\beta + 15\, \mathrm{e}^{-2\alpha\beta}]/(61440\alpha^4 a^4). \tag{4.22}$$

Note that both (4.16) and (4.21) approach zero exponentially fast for large m. They do not, however, vanish when the cycle frequency $\beta \to 0$. This is because there are still aperiodic anisotropies due to the factors $\mathrm{e}^{-\alpha\eta^2}$ and $\alpha^2/(\alpha^2+\eta^2)$ in (4.14) and (4.18) respectively.

It is of interest to examine whether anisotropic oscillations of the form (4.14) and (4.18) can be allowed in realistic cosmological models without producing so many particles as to be in conflict with observation. This is particularly relevant since it seems inevitable that the universe will emerge from the quantum gravity era, prior to the Planck time ($t_{\mathrm{P}} \simeq 10^{-43}$ s), with 'random' oscillations of this form. These oscillations will presumably be damped by a variety of mechanisms (see MacCallum (1979) for a review), including back reaction by the created particles. We are not in a position to study this back reaction, especially since vacuum polarisation effects are of first order in the anisotropic perturbation (see Hartle and Hu 1979, Horowitz 1979) and thus are of more importance to back reaction than the second-order particle production calculated here. We are, however, able to calculate roughly by which time oscillations of a given frequency must be damped if they are to produce less energy than is presently observed (we shall take this to be $\rho_0 = 10^{-30}$ gcm^{-3} as an upper bound).

We shall restrict our attention to oscillations about a radiation dominated Friedmann model with

$$a(t) = \gamma t^{1/2} = \tfrac{1}{2}\gamma^2\eta. \tag{4.23}$$

Consider first the oscillations (4.14), which can now be written in terms of 'cosmic' time t as

$$h_i(t) = \exp(-4\alpha t/\gamma^2)\cos(4\beta t/\gamma^2 + \delta_i), \tag{4.24}$$

and we observe that the frequency ν of oscillation is

$$\nu = 4\beta/\gamma^2, \tag{4.25}$$

while a measure of the isotropisation time $t_{\rm I}$ is given by

$$t_{\rm I} = \gamma^2/(4\alpha). \tag{4.26}$$

We simplify matters by considering only massless particles, which is certainly a good approximation for high frequencies, and write the energy density (4.17) at the present time $t_0 \simeq 10^{17}$ s in terms of ν and $t_{\rm I}$ as

$$\rho(t_0) = \frac{\hbar}{46080\pi c^5}\frac{t_{\rm I}^3}{t_0^2}(\nu^2 + t_{\rm I}^{-2})^{5/2}, \tag{4.27}$$

where we have reinstated factors of $\hbar$ and c that have previously been set equal to one.

For a given value of ν we wish to determine the time $t_{\rm I}$ at which $\rho(t_0) \simeq \rho_0$. Then, provided $\nu \gtrsim t_{\rm I}^{-1}$, if the oscillations damp within time $t_{\rm I}$ they are compatible with the observed energy density. Inserting the known constants in (4.27) and rearranging, we must thus solve

$$y^{10} + 5y^8 + 10y^6 + 10y^4 + 5y^2 + 1 \simeq 10^{176}(y/\nu)^4 \tag{4.28}$$

for $t_{\rm I}$, where $y \equiv \nu t_{\rm I}$.

If $y \simeq 1$ then this reduces to

$$\nu^{-4} \simeq 10^{-175},$$

so that $\nu \simeq 10^{44}\ {\rm s}^{-1}$ and $t_{\rm I} \simeq 10^{-44}\ {\rm s} \simeq t_{\rm P}$. This tells us that oscillations with Planck frequency must be damped within a few Planck times. As G does not enter the theory, this result would appear to be a consequence of one of the 'big number coincidences'.

If

$$1 \ll y \ll 10^{44}\nu^{-1}, \tag{4.29}$$

then to a good approximation (4.28) is

$$y^{10} \simeq 10^{176}(y/\nu)^4,$$

which implies $y \simeq 10^{29}\nu^{-2/3}$ and hence

$$t_{\rm I} \simeq 10^{29}\nu^{-5/3}, \tag{4.30}$$

applicable, because of (4.29), for $\nu \ll t_{\rm P}^{-1}$. Thus, for example, according to this criterion† there could still be oscillations with frequencies less than $10^7\ {\rm s}^{-1}$ present today.

† The observed energy density is not the only possible criterion (see for example Hu and Parker (1978) and references cited therein).

To obtain some idea of how model dependent are these results we turn now to the model with oscillations (4.18). Writing this in terms of cosmic time we have

$$h_i(t)=\frac{\alpha^2}{\alpha^2+4t/\gamma^2}\cos(2\beta t^{1/2}/\gamma+\delta_i), \tag{4.31}$$

and we see that a measure of the isotropisation time is given by

$$t_{\mathrm{I}}=\alpha^2\gamma^2/4, \tag{4.32}$$

while the frequency of oscillation slows down with time as

$$\nu(t)=\nu_{\mathrm{I}}(t_{\mathrm{I}}/t)^{1/2} \tag{4.33}$$

where ν_{I} is the frequency at the isotropisation time,

$$\nu_{\mathrm{I}}=2\beta/\gamma t_{\mathrm{I}}^{1/2}. \tag{4.34}$$

Writing the energy density of created, massless particles at the present time in terms of t_{I}, ν_{I} we have (from (4.22))

$$\rho(t_0)=\frac{\hbar}{983040c^5}\left(\frac{\nu_{\mathrm{I}}}{t_0}\right)^2\frac{1}{y^2}(4y^5+20y^3+3y+15\,\mathrm{e}^{-2y}), \tag{4.35}$$

where $y\equiv\nu_{\mathrm{I}}t_{\mathrm{I}}$. Inserting the known constants, the condition $\rho(t_0)\simeq\rho_0$ becomes

$$4y^5+20y^3+3y+15\,\mathrm{e}^{-2y}\simeq10^{88}(y/\nu_{\mathrm{I}})^2. \tag{4.36}$$

If $y\simeq1$ then this reduces to

$$\nu_{\mathrm{I}}^{-2}\simeq10^{-87}$$

so that $\nu_{\mathrm{I}}\simeq10^{-43}\,\mathrm{s}\simeq t_{\mathrm{P}}$ and $t_{\mathrm{I}}\simeq t_{\mathrm{P}}$. Thus, as in the previous example, Planck frequency oscillations must damp within a few Planck times.

If

$$1\ll y\ll10^{88}\nu_{\mathrm{I}}^{-2}, \tag{4.37}$$

then (4.36) can be approximated by

$$y^3\simeq10^{88}\nu_{\mathrm{I}}^{-2},$$

hence $y\simeq10^{29}\nu_{\mathrm{I}}^{-2/3}$ and thus, for $\nu_{\mathrm{I}}\ll t_{\mathrm{P}}^{-1}$,

$$t_{\mathrm{I}}\simeq10^{29}\nu_{\mathrm{I}}^{-5/3}, \tag{4.38}$$

in exact agreement with the result (4.30) of the previous model. We are thus led to believe that the results obtained here are probably quite insensitive to the model of oscillation.

5. Conclusion

We have shown that the perturbation method can be applied to numerous interesting situations, allowing fairly general conclusions to be drawn. In cases in which analytic results cannot be obtained, the general formulae that we have derived for the number and energy densities are particularly amenable to numerical calculation.

As yet, no one has obtained the complete stress tensor to second order in the anisotropic perturbation or in the mass. Hartle and Hu (1979) and Horowitz (1979) have obtained the vacuum expectation value of the massless stress tensor to first order in the anisotropic perturbations, thus obtaining the vacuum polarisation contribution. However, these contributions vanish in the out region and one must go to second order to obtain the particle production effects which persist in that region, giving the energy densities studied here. On the basis of our results, it seems that the calculation of the stress tensor to second order in the massive anisotropic case would be a most worthwhile, if herculean, task.

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Appendix

In obtaining (2.6) an integral of the following form is needed:

$$\begin{aligned} I &= \int_0^\infty \cos \alpha (k^2+c^2)^{1/2}\, \mathrm{d}k \\ &= \frac{\partial}{\partial\alpha}\int_0^\infty \frac{\sin \alpha (k^2+c^2)^{1/2}}{(k^2+c^2)^{1/2}}\, \mathrm{d}k \\ &= \int_0^\infty \mathrm{d}k \frac{\partial}{\partial\alpha}\int_0^\alpha \mathrm{d}x\, \mathrm{J}_0[c(\alpha^2-x^2)^{1/2}] \cos kx \\ &= \int_0^\infty \mathrm{d}k\left(\cos k\alpha + \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}x \frac{\partial}{\partial\alpha}\mathrm{J}_0[c(\alpha^2-x^2)^{1/2}] \cos kx\right) \\ &= \pi\delta(\alpha) + \frac{\pi}{2}\frac{\partial}{\partial\alpha}\mathrm{J}_0(c|\alpha|) \\ &= \pi\delta(\alpha) + \frac{\pi}{2}\frac{\partial}{\partial\alpha}\mathrm{J}_0(c\alpha) \\ &= \pi\delta(\alpha) - \frac{\pi}{2}c\mathrm{J}_1(c\alpha). \end{aligned} \tag{A1}$$

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